Topic 2 – Linear first order ODEs

A linear first order ODE is an equation of the form

$$a_1(x)y' + a_0(x)y = g(x)$$

If we are considering an interval I where $a_1(x) \neq 0$ for any x in I then we can divide through by $a_1(x)$ to get

$$y' + a(x)y = b(x)$$

where $a(x) = \dfrac{a_0(x)}{a_1(x)}$ and $b(x) = \dfrac{g(x)}{a_1(x)}$.

This is the type of equation that we will consider for now.

Suppose we have a linear first order ODE of the form

$$\boxed{y' + a(x)y = b(x)} \quad (*)$$

where $a(x)$ and $b(x)$ are continuous on an open interval I.

Ex:

$y' + \underbrace{2x}_{a(x)=2x} y = \underbrace{x}_{b(x)=x}$

$I = (-\infty, \infty)$

Let's solve this.

Suppose $\phi(x)$ solves $(*)$ on I.

That is,

$$\boxed{\phi'(x) + a(x)\phi(x) = b(x)} \quad (**)$$

for all x in I.

Let $A(x)$ be an anti-derivative of $a(x)$ on I.

Ex:

$a(x) = 2x$

$A(x) = x^2$

Note: $A(x)$ exists by the FTOC since $a(x)$ is continuous.

Multiply (**) by $e^{A(x)}$ to get:

$$e^{A(x)}\phi'(x) + a(x)\phi(x) = e^{A(x)} b(x)$$

This gives (by the product rule $(fg)' = f'g + g'f$)

$$\left(e^{A(x)}\phi(x)\right)' = e^{A(x)} b(x)$$

integrate both sides w/ respect to x

Let $B(x)$ be an anti-derivative of $e^{A(x)} b(x)$ on I.

Then ϕ solves (*) on I if and only if

$$e^{A(x)}\phi(x) = B(x) + C$$

where C is some constant.

So, ϕ solves (*) on I if and only if

$$\phi(x) = B(x) e^{-A(x)} + C e^{-A(x)}$$

Since all the steps above are reversable since $e^{A(x)} \neq 0$ we know we have found the general solution to $(*)$.

$\blacksquare$

Ex: Solve

$$y' + \underbrace{2x}_{a(x)=2x} y = \underbrace{x}_{b(x)=x}$$

on $I = (-\infty, \infty)$.

Step 1: $A(x) = \int 2x\,dx = x^2$

Multiply by $e^{A(x)} = e^{x^2}$ to get

$$e^{x^2} y'(x) + 2x e^{x^2} y(x) = x e^{x^2}$$

Step 2: Undo the product rule on the left-hand side:

$$\left(e^{x^2} \cdot y(x) \right)' = x e^{x^2}$$

Step 3: Integrate both sides to get:

$$e^{x^2}\cdot y(x) = \underbrace{\frac{1}{2}e^{x^2} + C}$$

$$\int x e^{x^2}\,dx = \frac{1}{2}\int e^{u}\,du = \frac{1}{2}e^{u} + C = \frac{1}{2}e^{x^2} + C$$

$$u = x^2$$
$$du = 2x\,dx$$
$$\frac{1}{2}du = x\,dx$$

Step 4:

Thus, $y = \frac{1}{2} + Ce^{-x^2}$

for some constant C.

$\blacksquare$

Ex: Let's solve

$$y' + \underbrace{\cos(x)}_{a(x)=\cos(x)} y = \underbrace{\sin(x)\cos(x)}_{b(x)=\sin(x)\cos(x)}$$

on $I = (-\infty, \infty)$

Step 1: Let $A(x) = \sin(x)$.

Then $A'(x) = \cos(x) = a(x)$

Multiply be $e^{A(x)} = e^{\sin(x)}$.

We get

$$e^{\sin(x)} y'(x) + \cos(x) e^{\sin(x)} y(x) = \sin(x)\cos(x) e^{\sin(x)}$$

Step 2: Undo the product rule on the left side to get:

$$\left(e^{\sin(x)} y(x) \right)' = \sin(x)\cos(x) e^{\sin(x)}$$

Step 3: Integrate both sides.

$$e^{\sin(x)}\, y(x) = \sin(x)e^{\sin(x)} - e^{\sin(x)} + C$$

$$\int \sin(x)\cos(x)\, e^{\sin(x)}\, dx$$

$$= \int t e^{t}\, dt = t e^{t} - \int e^{t}\, dt$$

$t = \sin(x)$
$dt = \cos(x)\, dx$

$u = t \qquad dv = e^{t}\, dt$
$du = dt \qquad v = e^{t}$

$\int u\, dv = uv - \int v\, du$

$$= t e^{t} - e^{t} + C$$

$$= \sin(x)e^{\sin(x)} - e^{\sin(x)} + C$$

Step 4: Thus,

$$y = \sin(x) - 1 + Ce^{-\sin(x)}$$

where C is some constant.

Ex: Consider the equation

$$xy' + y = 3x^3 + 1$$

on $I = (0, \infty)$.

Since $x \neq 0$ on I we can divide by x to get

$$y' + \underbrace{\frac{1}{x}}_{a(x)} y = \underbrace{3x^2 + \frac{1}{x}}_{b(x)}$$

Step 1: Let $A(x) = \ln(x)$.
Then, $A'(x) = \frac{1}{x}$ for all x in I.
Multiply by $e^{A(x)} = e^{\ln(x)} = x$ to get

$$xy' + y = 3x^3 + 1$$

Step 2: Undo the product rule to get

$$(xy)' = 3x^3 + 1$$

Step 3: Integrate both sides to get

$$xy = \int (3x^3 + 1)\,dx = \frac{3}{4}x^4 + x + C$$

Step 4: Thus,

$$y = \frac{3}{4}x^3 + 1 + \frac{C}{x}$$

where C is a constant.

Ex: Solve

$$xy' + y = 3x^3 + 1$$
$$y(1) = 2$$

on $I = (0, \infty)$

From above we know $y = \frac{3}{4}x^3 + 1 + \frac{c}{x}$.

Plugging in $y(1) = 2$ we get

$$2 = y(1) = \frac{3}{4}(1)^3 + 1 + \frac{c}{1}$$

So,

$$2 = \frac{7}{4} + c$$

Thus,

$$c = \frac{1}{4}$$

Thus,

$$y = \frac{3}{4}x^3 + 1 + \frac{c}{x}.$$